

A CRITERION FOR THE LEGENDRIAN SIMPLICITY OF THE CONNECTED SUM

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ABSTRACT. In this paper, we provide the necessary and sufficient conditions for the connected sum of knots in S^3 to be Legendrian simple.

1. INTRODUCTION

Throughout this paper, we consider oriented Legendrian knots in S^3 with the standard tight contact structure ξ_{std} .

A Legendrian knot theory is the study of Legendrian knots up to Legendrian isotopy which is much more restricted compared to the smooth or piecewise-linear isotopy in the classical knot theory. More precisely, by considering the ambient space as the one-point compactification of \mathbb{R}^3 and taking a suitable projection, these two theories are differ by whether *positive and negative (de)stabilizations* are allowed or not. Therefore for given topological knot type K , there are infinitely many inequivalent Legendrian isotopy classes of topological knot type K , and we denote by $\mathcal{L}(K)$ the set of Legendrian isotopy classes of K . There are two well-known Legendrian knot invariants, *Thurston-Bennequin number* $tb(L)$ and *rotation number* $r(L)$ that can be used to classify Legendrian knots in $\mathcal{L}(K)$, and they are called *classical invariants*. Refer to [8] for details. Then a topological knot type K is said to be *Legendrian simple* if Legendrian knots in $\mathcal{L}(K)$ are classified by the classical invariants.

The Legendrian simplicity for unknot has been shown by Eliashberg and Fraser in 1995 [3], and for figure-eight and torus knots by Etnyre and Honda in 2001 [4]. Especially, Etnyre and Honda in their followed paper [5] provided a complete combinatorial description for the connected sum of Legendrian knots, and proved that the Legendrian simplicity is not closed under the connected sum.

On the other hand, they also proved in [6] that the cabling operation preserves the Legendrian simplicity under the *uniform thickness property* (UTP). The important benefits of (UTP) are that it will be preserved not only by cabling with the certain condition, but also by the connected sum. Hence, if started with Legendrian simple and UTP knot, one may produce the arbitrarily many Legendrian simple knots by the iterated cabling as studied in [7, 9, 10]. However, as seen above, the Legendrian simplicity may not be preserved under connected sum, and therefore the attempt of cabling after connected sum may fail.

This paper provides the necessary and sufficient conditions for the connected sum to be Legendrian simple as follows.

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Theorem 1. *Let $K = (\#^{a_1} K_1) \# \dots (\#^{a_n} K_n)$ be a connected sum decomposition of K with pairwise distinct prime knots. Then K is Legendrian simple if and only if all K_i 's are Legendrian simple and one of the following is satisfied:*

- (1) $|Peak(K_i)| = 1$ for all i ;
- (2) There exists only one i such that $|Peak(K_i)| = 2$, $a_i \geq 2$ and $|Peak(K_j)| = 1$ if $j \neq i$;
- (3) There exists only one i such that $|Peak(K_i)| \geq 3$, $a_i = 1$ and $|Peak(K_j)| = 1$ if $j \neq i$.

Here, $Peak(K_i)$ is the set of Legendrian knots in $\mathcal{L}(K_i)$ which can not be destabilized in either ways.

Therefore this theorem provides the way to produce the new Legendrian simple knots. Moreover, this can be used with the the cabling construction, as well.

The rest of this paper consists of the following. In section 2, we introduce the basic notions and briefly review the known results about Legendrian connected sum. In section 3, we prove the main result.

2. PRELIMINARIES

2.1. Basic notions. A *topological oriented knot*, simply a *knot* K from now on, is a smooth embedding of S^1 into S^3 , and a *knot type* $[K]$ is a smooth isotopy class of K . The natural orientation of K comes from $d\theta$ where S^1 is parametrized by θ . A knot K is *trivial* or an *unknot* if K bounds an embedded disc in S^3 .

A *Legendrian knot* L is a Legendrian embedding of S^1 into (S^3, ξ_{std}) , that is, L is everywhere tangent to the contact plane ξ_{std} . Here ξ_{std} is the standard contact structure on S^3 . Similarly, a *Legendrian knot type* $[L]$ is the Legendrian isotopy class of L . From now on, we simply use K, L instead of $[K], [L]$ to denote knot types unless any ambiguity occurs.

For given knot type K , we denote by $\mathcal{L}(K)$ the set of Legendrian knots of topological knot type K . Then as mentioned before, any two elements in $\mathcal{L}(K)$ can be connected with a sequence of two special types of isotopies, called *positive and negative stabilizations* S_{\pm} , and their inverses. Note that these two stabilizations are commutative. In other words, $S_+(S_-(L)) = S_-(S_+(L))$ for any Legendrian knot L . In the diagrammatic viewpoint, S_{\pm} is as depicted in Figure 1. Hence we can regard $\mathcal{L}(K)$ as a connected, directed graph by adding directed edges $(L, S_+(L))$ and $(L, S_-(L))$ for each $L \in \mathcal{L}(K)$.

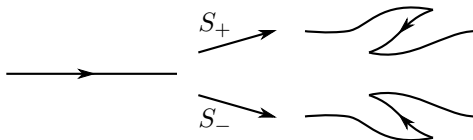


FIGURE 1. Positive and negative stabilization S_{\pm} in the front projection

The classical invariants, Thurston-Bennequin number $tb(L)$ and the rotation number $r(L)$, for a Legendrian knot L change as follows under stabilizations.

$$tb(S_{\pm}(L)) = tb(L) - 1, \quad r(S_{\pm}(L)) = r(L) \pm 1.$$

This implies that $\mathcal{L}(K)$ has no directed loop different from constant, and admits a graded poset structure $(\mathcal{L}(K), \prec)$ by declaring $S_{\pm}(L) \prec L$, whose grading is given by tb . We say that L is a *parent* of both $S_+(L)$ and $S_-(L)$.

On the other hand, there is another important poset $\mathcal{M}(K)$, called *mountain range* defined by the range $\Phi(\mathcal{L}(K))$ of the pair $\Phi = (tb, r) : \mathcal{L}(K) \rightarrow \mathbb{Z}^2$ of classical invariants. We will follow the common convention of regarding tb and r as vertical and horizontal axes, respectively, and the words, such as *left*, *right* and so on, have the suitable meaning according to this convention. Especially, *maximal* means having maximal tb among elements of given (subset of) $\mathcal{L}(K)$ or $\mathcal{M}(K)$.

Note that $\mathcal{M}(K)$ is never bounded below because tb can be decreased arbitrarily by taking stabilizations in $\mathcal{L}(K)$. However, Bennequin in [1] showed that $\mathcal{M}(K)$ is always bounded above as follows.

Theorem 2. [1] *For given knot type K and $L \in \mathcal{L}(K)$,*

$$tb(L) + |r(L)| \leq 2g(K) - 1,$$

where $g(K)$ is a genus of K .

This result has been improved in many ways, related to classical link invariants such as genus [1, 15], polynomials [11, 14], or other invariants such as Khovanov homology [12], knot Floer homology [13] and so on. However they are not essential in this paper and we omit the detail.

Though Φ is not injective in general, we draw $\mathcal{L}(K)$ on the (tb, r) -plane via Φ by perturbing edges and vertices slightly if necessary, as seen in Figure 2.

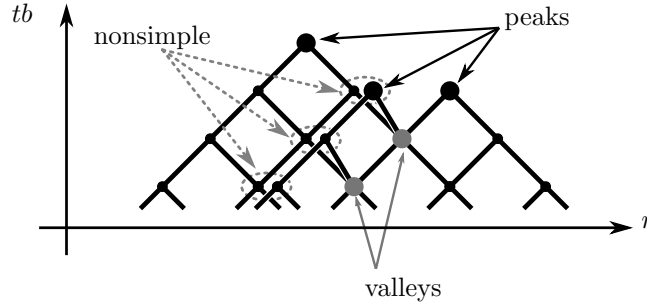


FIGURE 2. *Peak and Valley* for a poset

Then we say that K is *Legendrian simple* if Φ is injective. The simplicity can be defined pointwise as well, according to the number of the inverse image under Φ . In other words, a point $p \in \mathcal{M}(K)$ is *simple* if p has only one inverse under Φ , and p is *nonsimple* otherwise.

For a given poset \mathcal{P} , we denote by $Peak(\mathcal{P})$ the set of elements of \mathcal{P} having no parent, and by $Valley(\mathcal{P})$ the set of elements $V \in \mathcal{P}$ such that V possesses two parents which have no common parent. A possible example of *Peak* and *Valley* for a poset is depicted in Figure 2. We will consider $Peak(\mathcal{L}(K))$, $Valley(\mathcal{L}(K))$ as well as $Peak(\mathcal{M}(K))$, $Valley(\mathcal{M}(K))$.

If K is Legendrian simple, then $\mathcal{L}(K)$ and $\mathcal{M}(K)$ are isomorphic as posets, and we consider *Peak* and *Valley* only for $\mathcal{L}(K)$ and denote by $Peak(K)$ and $Valley(K)$. One of the obvious observation for a Legendrian simple knot K is that

$$|Peak(K)| = |Valley(K)| + 1 < \infty.$$

On the other hand, if K is not Legendrian simple, then we can choose a point N_{max} which is maximal among nonsimple points in $\mathcal{M}(K)$. Observe that N_{max} may not be unique, and any point above N_{max} in $\mathcal{M}(K)$ is simple by definition.

Lemma 3. *Let K be a Legendrian nonsimple knot and N_{max} be as above. Then either*

- (1) $\Phi^{-1}(N_{max}) \cap \text{Peak}(\mathcal{L}(K)) \neq \emptyset$ or
- (2) $|\Phi^{-1}(N_{max})| = 2$ and $N_{max} \in \text{Valley}(\mathcal{M}(K))$.

Proof. Let $\Phi^{-1}(N_{max}) = \{L_1, \dots, L_k\}$. Since N_{max} is not simple, $k \geq 2$.

Suppose $L_i \notin \text{Peak}(\mathcal{L}(K))$ for all i . Then all L_i 's have at least 1 parent in $\mathcal{L}(K)$ and all these parents become parents of N_{max} in $\mathcal{M}(K)$ via Φ . Since N_{max} has at most 2 parents in $\mathcal{M}(K)$ and all points above N_{max} are simple, there are at most 2 Legendrian knots which are parents of L_i 's. Hence $\Phi^{-1}(N_{max})$ consists of exactly two Legendrian knots where each has only 1 parent, and therefore $N_{max} \in \text{Valley}(\mathcal{M}(K))$. \square

Examples of possible local pictures near N_{max} are depicted in Figure 3. The left two are involving at least 1 peak but the right one corresponds to a valley in $\mathcal{M}(K)$.

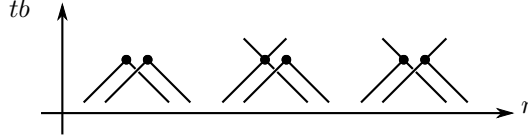


FIGURE 3. Local pictures of $\mathcal{L}(K)$ near N_{max}

2.2. Connected sums. Let K_1, K_2 be two knots in S^3 . Roughly speaking, a *connected sum* $K_1 \# K_2 \subset S^3$ is a knot obtained by gluing $K_1 \setminus \alpha_1$ and $K_2 \setminus \alpha_2$ along the oriented end points, where α_i 's are subarcs of K_i unknotted in small enough 3 balls $B_i \subset S^3$. It is equivalent to the connected sum of pairs (S^3, K_1) and (S^3, K_2) . For convenience sake, we denote by $\#^n K$ the connected sum of n -copies of K .

A nontrivial knot K is said to be *prime* if $K = K_1 \# K_2$ implies that one of K_i 's is trivial. If K is neither trivial nor prime, we say that K is *composite*. Then for any nontrivial knot K , there is a unique prime decomposition $K = (\#^{a_1} K_1) \# \dots (\#^{a_n} K_n)$ up to permuting K_i 's, where $a_i \geq 1$ and all K_i 's are pairwise different prime knots. For two knots K_1 and K_2 , we say that K_1 and K_2 are *relatively prime* unless there is nontrivial common connected summand.

To make this concepts fit into the Legendrian knot theory, we need the following.

Theorem 4. [2] *Given two 3-manifolds M_1, M_2 there is an isomorphism*

$$\pi_0(\text{Tight}(M_1)) \times \pi_0(\text{Tight}(M_2)) \xrightarrow{\sim} \pi_0(\text{Tight}(M_1 \# M_2)),$$

where $\pi_0(\text{Tight}(M_i))$ is the set of contact structures on M_i up to contact isotopy.

Since we consider S^3 as the ambient space which has the unique contact structure ξ_{std} up to contact isotopy, there is no ambiguity at all. Therefore, we may define $L_1 \# L_2$ in (S^3, ξ_{std}) by the connected sum of pairs as before for given two Legendrian knots L_1, L_2 in (S^3, ξ_{std}) . Figure 4 shows a pictorial definition of the connected sum of (Legendrian) knots.

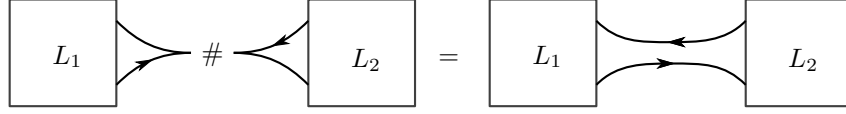


FIGURE 4. A pictorial definition of the connected sum

For given two knots K_1, K_2 and Legendrian knots $L_i \in \mathcal{L}(K_i)$, Etnyre and Honda in [5] showed not only well-definedness of $L_1 \# L_2$ but also a complete description of the relation between $\mathcal{L}(K_1 \# K_2)$ and $\mathcal{L}(K_i)$'s for knots in arbitrary 3-manifolds with tight contact structures. Here we introduce their results only for knots in S^3 . We denote by \mathbf{S}_n the symmetric group on $\{1, \dots, n\}$.

Theorem 5. [5] *Let $K = K_1 \# K_2 \# \dots \# K_n$ be a prime decomposition of a knot K in S^3 . Then the map*

$$C : (\mathcal{L}(K_1) \times \dots \times \mathcal{L}(K_n) / \sim) \rightarrow \mathcal{L}(K_1 \# \dots \# K_n)$$

given by $(L_1, \dots, L_n) \mapsto L_1 \# \dots \# L_n$ is a bijection. Here the equivalence relation \sim is of two types:

- (1) $(L_1, \dots, S_{\pm}(L_i), L_{i+1}, \dots, L_n) \sim (L_1, \dots, L_i, S_{\pm}(L_{i+1}), \dots, L_n)$,
- (2) $(L_1, \dots, L_n) \sim (L_{\sigma(1)}, \dots, L_{\sigma(n)})$, where $\sigma \in \mathbf{S}_n$ such that $K_{\sigma(i)}$ is isotopic to K_i .

The behavior of classical invariants under the connected sum is quite obvious as follows.

Lemma 6. *Let L_1, L_2 be two Legendrian knots. Then*

$$tb(L_1 \# L_2) = tb(L_1) + tb(L_2) + 1, \quad r(L_1 \# L_2) = r(L_1) + r(L_2).$$

For a topological space X , a *symmetric product* $\text{Sym}^n(X)$ of X is defined by the quotient $\prod^n X / \mathbf{S}_n$ under the obvious action of the symmetric group. We denote an equivalent class of (x_1, \dots, x_n) (or a set with repetition) in $\text{Sym}^n(X)$ by $[x_1, \dots, x_n]$. Then the direct consequences of the above theorem are as follows.

Corollary 7. *For a prime K , then*

$$C : \text{Sym}^n(\text{Peak}(\mathcal{L}(K))) \rightarrow \text{Peak}(\mathcal{L}(\#^n K))$$

is bijective.

Proof. It is obvious that type (2) equivalence relation is always applicable for any $\sigma \in \mathbf{S}_n$ but type (1) is never applicable on $\prod^n \text{Peak}(\mathcal{L}(K))$ because any element in $\text{Peak}(\mathcal{L}(K))$ can not be destabilized. Hence C is well-defined on $\text{Sym}^n(\text{Peak}(\mathcal{L}(K)))$. Moreover, by definition of the connected sum, C maps bijectively onto $\text{Peak}(\mathcal{L}(\#^n K))$. \square

Corollary 8. *For two relatively prime knots K_1 and K_2 ,*

$$C : \text{Peak}(\mathcal{L}(K_1)) \times \text{Peak}(\mathcal{L}(K_2)) \rightarrow \text{Peak}(\mathcal{L}(K_1 \# K_2))$$

is bijective.

Proof. Since K_1 and K_2 are relatively prime, the equivalence relation \sim does not change two summand. Moreover, by the same reason as above, type (1) equivalence relation is not applicable either. \square

- (1) $L_i = L_{i-1}$ if $\epsilon_i = 0$;
- (2) $L_i = S_{\epsilon_i}(L_{i-1})$ if $\epsilon \neq 0, \eta_i = 1$; and
- (3) $L_{i-1} = S_{\epsilon_i}(L_i)$ if $\epsilon \neq 0, \eta_i = -1$.

We define the *reverse* $\bar{\gamma} = S_{\epsilon_k}^{-\eta_k} \dots S_{\epsilon_1}^{-\eta_1}$ of γ by changing all exponents. Note that this is different from the usual inverse. Geometrically, for any mountain ranges that γ and $\bar{\gamma}$ are realized, $\bar{\gamma}$ goes down or right if γ goes up or left, respectively, and *vice versa* because all exponents are reversed. See Figure 5 for example. This operation plays an important role for describing the equivalence relation \sim as follows.

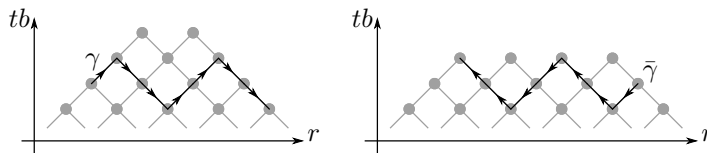
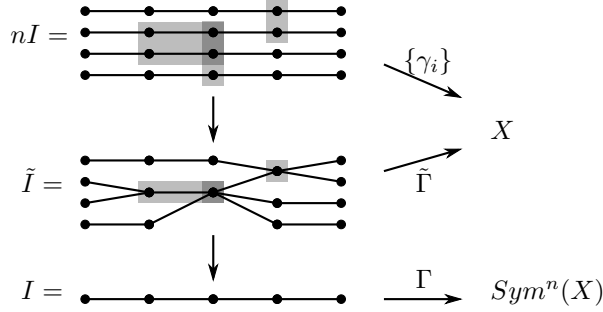


FIGURE 5. A path $\gamma = S_+^2 S_-^{-2} S_+^2 S_-^{-1}$ and its reverse $\bar{\gamma} = S_+^{-2} S_-^2 S_+^{-2} S_-$

- (1) *there exists a permutation $\sigma \in \mathbf{S}_n$ such that $L'_{\sigma(i)} \in \gamma_i(L_i)$,*
- (2) *for each $\ell \leq k$, ℓ -th words of γ_i 's are either $[S_+, S_+^{-1}, S_0, \dots, S_0]$ or $[S_-, S_-^{-1}, S_0, \dots, S_0]$.*

We can resolve the branch locus of \tilde{I} to obtain n disjoint intervals nI and n paths $\{\gamma_i\}$ of length k in $\mathcal{L}(K)$ such that each joins some of L_i and L'_j . Note that this process is not unique, but the resulting paths satisfy the condition above by definition of the connected sum. Therefore the existence of paths is the same as the equivalence with respect to \sim . \square

In particular, we can conclude that $L_1 \# L_2$ is different from $L'_1 \# L'_2$ if there is no such path.

FIGURE 6. Multi-paths $\{\gamma_i\}$ describing a path $\Gamma : I \rightarrow \text{Sym}^n(X)$ space.

Proof. Since K_1 and K_2 are relatively prime, the equivalence relation by permuting components is never applicable. Hence we only need to consider the first type of equivalence relation in Theorem 5.

However, the generator for the equivalence relation of the first type obviously defines a path of length 1 which satisfies the assumption. Moreover, realizable paths are closed under concatenations whenever their ends match, and therefore the lemma follows. \square

3. MAIN RESULTS

To prove Theorem 1 we will show the following propositions.

Proposition 11. *Let K be a prime knot. If $\#^n K$ is Legendrian simple, then so is K .*

Proposition 12. *Let K_1, K_2 be two relatively prime knots. If $K_1 \# K_2$ is Legendrian simple, then so are K_1 and K_2 .*

Hence for the connected sum to be Legendrian simple, each of its summands must be Legendrian simple. However, even for Legendrian simple knots, their connected sum need not be Legendrian simple when they have many peaks as follows.

Proposition 13. *Let K be a prime and Legendrian simple knot. Then $\#^n K$ is Legendrian simple for $n \geq 2$ if and only if $|\text{Peak}(K)| \leq 2$.*

Moreover,

$$|\text{Peak}(\#^n K)| = \begin{cases} 1 & \text{if } |\text{Peak}(K)| = 1; \\ n + 1 & \text{if } |\text{Peak}(K)| = 2. \end{cases}$$

Proposition 14. *Let K_1 and K_2 be relatively prime and Legendrian simple knots. Then $K_1 \# K_2$ is Legendrian simple if and only if either $|\text{Peak}(K_1)| = 1$ or $|\text{Peak}(K_2)| = 1$. In this case,*

$$|\text{Peak}(K_1 \# K_2)| = |\text{Peak}(K_1)| \cdot |\text{Peak}(K_2)|.$$

Then Theorem 1 is nothing but the reorganization of the propositions above.

Now we prove the propositions.

Proof of Proposition 11. Let $L \in \text{Peak}(\mathcal{L}(K))$ be a maximal element, and suppose that K is Legendrian nonsimple but $\#^n K$ is Legendrian simple. Then we can choose $N_{\max} \in \mathcal{N}(K)$ as before, and there are two cases as follows by Lemma 3.

If $\Phi^{-1}(N_{max}) \cap Peak(\mathcal{L}(K_1)) \neq \emptyset$, then there are two different Legendrian knots $L_1, L'_1 \in \Phi^{-1}(N_{max})$ such that $L_1 \in Peak(\mathcal{L}(K))$. Since $L \in Peak(\mathcal{L}(K))$ as well, $L_1 \# (\#^{n-1}L) \in Peak(\mathcal{L}(\#^n K))$ by Corollary 7. If $L'_1 \notin Peak(\mathcal{L}(K))$, then $L'_1 \# (\#^{n-1}L) \notin Peak(\mathcal{L}(\#^n K))$. Therefore $L_1 \# (\#^{n-1}L)$ and $L'_1 \# (\#^{n-1}L)$ are different. Otherwise, if $L'_1 \in Peak(\mathcal{L}(K))$, then $L_1 \# (\#^{n-1}L)$ and $L'_1 \# (\#^{n-1}L)$ are still different since $[L_1, L, \dots, L] \neq [L'_1, L, \dots, L]$ and by Corollary 7.

If $\Phi^{-1}(N_{max}) \cap Peak(\mathcal{L}(K)) = \emptyset$, then $\Phi^{-1}(N_{max}) = \{L_1, L'_1\}$ by lemma 3. Suppose $L_1 \# (\#^{n-1}L)$ and $L'_1 \# (\#^{n-1}L)$ are equivalent. Then by Lemma 9, we may assume that there are paths $\gamma_1, \dots, \gamma_n$ such that γ_1 is realizable at L_1 . Note that all γ_i 's are lying above N_{max} except for end points if we take Φ . Since all points above N_{max} are simple, we may identify $\mathcal{L}(K)$ and $\mathcal{M}(K)$ above N_{max} .

If $L'_1 \notin \gamma_1(L_1)$, then $L \in \gamma_1(L_1)$ and $L'_1 \in \gamma_i(L)$ for some $i \neq 1$. This implies that both L_1 and L'_1 can be joined with L above N_{max} but this is impossible because parents of L_1 and L'_1 are lying in the different connected components of the region above N_{max} .

On the other hand, if $L'_1 \in \gamma_1(L_1)$ then by the exactly same reason about the parents of L_1 and L'_1 , this is impossible too.

In all cases, $L_1 \# (\#^{n-1}L)$ and $L'_1 \# (\#^{n-1}L)$ are different but it is obvious that they share the classical invariants. Therefore this contradicts to the Legendrian simplicity of $\#^n K$. \square

Proof of Proposition 12. The proof is essentially same as the previous one by using Corollary 8 and Lemma 10 instead of Corollary 7 and Lemma 9. \square

Proof of Proposition 13. Recall that $|Peak(K)| = |Valley(K)| + 1$ for a Legendrian simple knot K .

Suppose $|Peak(K)| \geq 3$. Then there are at least two Legendrian knots V_1, V_2 lying in $Valley(K)$. We may assume that $r(V_1) < r(V_2)$. For each V_i , there are two parents L_i, L'_i so that $tb(L_i) = tb(L'_i) = tb(V_i) + 1$, and $r(V_1) = r(L_1) + 1 = r(L'_1) - 1$, $r(V_2) = r(L_2) - 1 = r(L'_2) + 1$. In addition, we fix a maximal $L_3 \in Peak(K)$ as depicted in Figure 7.

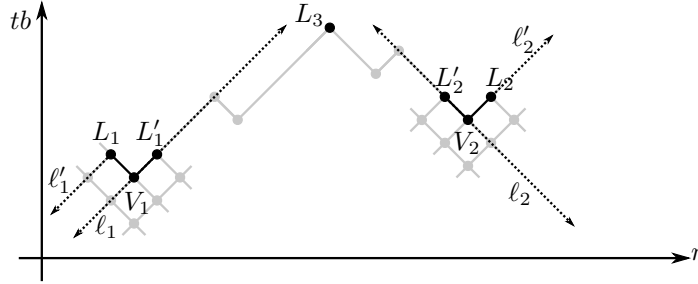


FIGURE 7. Two valleys V_i , hillsides ℓ_i , parents L_i, L'_i and chosen peak L_3

Suppose $L_1 \# L_2 \# (\#^{n-2}L_3)$ and $L'_1 \# L'_2 \# (\#^{n-2}L_3)$ are same in $\mathcal{L}(\#^n K)$. Then there are paths $\gamma_1, \dots, \gamma_n$ as before. We may assume that γ_1 and γ_2 are realizable at L_1 and L_2 , respectively. For each valley V_i , a *hillside* ℓ_i is the line defined by

$$\ell_1 : tb - r = tb(V_1) - r(V_1), \quad \ell_2 : tb + r = tb(V_2) + r(V_2).$$

We claim that each γ_i never hit the hillside ℓ_i .

Suppose not for γ_1 . Then the last move just before hitting ℓ_1 must correspond to S_+ from a point in the ray

$$\ell'_1 : tb - r = tb(L_1) - r(L_1), \quad tb \leq tb(L_1).$$

At that time, since γ_1 lies at the same level of the initial position or below and there is no point above L_3 , the only possibility is that γ_2 must lie on the another hillside

$$\ell'_2 : tb - r = tb(L_2) - r(L_2)$$

at the same level of L_2 or above.

By Lemma 9, the corresponding move in γ_2 is S_+^{-1} but it can not be performed since there is no point above ℓ'_2 . This is a contradiction. Similarly γ_2 never hit the hillside ℓ_2 , and the claim is proved.

This claim implies that $\gamma_1(L_1)$ and $\gamma_2(L'_2)$ are separated by 2 lines ℓ_1 and ℓ_2 , and therefore $L'_1, L_2 \notin \gamma_1(L_1)$ and $L_1, L_2 \notin \gamma_2(L'_2)$.

The only possibility is that $L_3 \in \gamma_1(L_1) \cap \gamma_2(L'_2)$.

$L'_1 \notin \gamma_1(L_1)$ and $L_2 \notin \gamma_2(L'_2)$, and therefore only possibility is that $L_3 \in \gamma_1(L_1) \cap \gamma_2(L'_2)$. However this is not possible either because the regions where γ_i 's are lying are separated by lines ℓ_1, ℓ_2 . Therefore $L_1 \# L'_2 \# (\#^{n-2} L_3)$ and $L'_1 \# L_2 \# (\#^{n-2} L_3)$ are different but share the classical invariants. Hence $\#^n K$ is not Legendrian simple.

Suppose $|Peak(K)| = 1$, that is, $\mathcal{L}(K)$ has the greatest element L and all Legendrian knots in $\mathcal{L}(K)$ are of the form $S_+^a S_-^b(L)$. Moreover, these stabilizations can be relocated freely among the connected summands. Hence any Legendrian knot in $\mathcal{L}(\#^n K)$ is equivalent to $S_+^a S_-^b(L) \# (\#^{n-1} L)$ for some $a, b \geq 0$, and its tb and r determine a and b uniquely. Therefore $\#^n K$ is Legendrian simple.

Finally, suppose $Peak(K) = \{P_1, P_2\}$ and $Valley(K) = \{V\}$. Without loss of generality, we may assume that $r(P_1) < r(V) < r(P_2)$. Then

$$S_+^{-r'(P_1)}(P_1) = V = S_-^{r'(P_2)}(P_2), \text{ and } tb'(P_1) = -r'(P_1), tb'(P_2) = r'(P_2)$$

where $r'(L) = r(L) - r(V)$ and $tb'(L) = tb(L) - tb(V)$. See Figure 8.

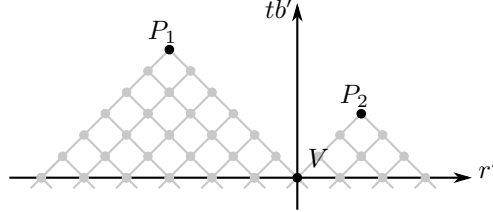


FIGURE 8. Mountain range with exactly 2 peaks

As before, all Legendrian knots in $\mathcal{L}(K)$ are of the form either $S_+^a S_-^b(P_1)$ or $S_+^a S_-^b(P_2)$, and any Legendrian knot L in $\mathcal{L}(\#^n K)$ is equivalent to $S_+^a S_-^b((\#^p P_1) \# (\#^q P_2))$ for $p + q = n$. We simply denote this by $L(a, b, p, q)$. Then for $L = L(a, b, p, q)$,

$$tb(L) = p \cdot tb(P_1) + q \cdot tb(P_2) + (n - 1) - a - b,$$

$$r(L) = p \cdot r(P_1) + q \cdot r(P_2) + a - b.$$

We consider two invariants defined by using tb and r as follows.

$$X(L) = \frac{1}{2} (tb(L) + r(L) - n(tb(V) + r(V)) - (n - 1)) = q \cdot r'(P_2) - b,$$

$$Y(L) = -\frac{1}{2}(tb(L) - r(L) - n(tb(V) - r(V)) - (n-1)) = p \cdot r'(P_1) + a.$$

Now suppose two Legendrian knots $L = L(a, b, p, q)$ and $L' = L(a', b', p', q')$ share the same tb and r . Then they also share X and Y , and so

$$X(L) - X(L') = (q - q')r'(P_2) - (b - b') = 0,$$

$$Y(L) - Y(L') = (p - p')r'(P_1) + (a - a') = 0.$$

If $b = b'$ or $a = a'$ then $p = p'$ and $q = q'$ since neither $r'(P_1)$ nor $r'(P_2)$ vanishes.

If $b > b'$, then $q > q'$, $p < p'$, and

$$a' = a + (p - p')r'(P_1), \quad b = (q - q')r'(P_2) + b'.$$

Moreover,

$$\begin{aligned} L(a, b, p, q) &= S_+^a S_-^b ((\#^p P_1) \# (\#^q P_2)) \\ &= S_+^a S_-^{b'} ((\#^p P_1) \# (\#^{q'} P_2) \# (\#^{q-q'} S_-^{r'(P_2)} P_2)) \\ &= S_+^a S_-^{b'} ((\#^p P_1) \# (\#^{q'} P_2) \# (\#^{q-q'} V)) \\ &= S_+^a S_-^{b'} ((\#^p P_1) \# (\#^{q'} P_2) \# (\#^{p'-p} S_+^{-r'(P_1)} P_1)) \\ &= S_+^{a+(p-p')r'(P_1)} S_-^{b'} ((\#^{p'} P_1) \# (\#^{q'} P_2)) \\ &= S_+^{a'} S_-^{b'} ((\#^{p'} P_1) \# (\#^{q'} P_2)) = L(a', b', p', q'). \end{aligned}$$

Conversely, the same result holds for $b < b'$ by changing the roles of a, b, p, q and a', b', p', q' . Therefore the classical invariants determine exactly one Legendrian knot in $\mathcal{L}(\#^n K)$, and so $\#^n K$ is Legendrian simple.

The number of peaks directly follows from Corollary 7. \square

Proof of Proposition 14. Suppose $|Peak(K_i)| \geq 2$, or $|Valley(K_i)| \geq 1$ for all i . Let $V_i \in Valley(K_i)$ and L_i, L'_i be two stabilizations of V_i such that $r(V_i) = r(L_i) + 1 = r(L'_i) - 1$. Then $L_1 \# L'_2$ and $L'_1 \# L_2$ are different by the essentially same argument as before and by Lemma 10. Therefore $K_1 \# K_2$ is not Legendrian simple.

Suppose $|Peak(K_1)| = 1$, and let L be the unique maximal element of $\mathcal{L}(K_1)$. Then as before, all Legendrian knots in $\mathcal{L}(K_1)$ are of the form $S_+^a S_-^b(L)$, and so any Legendrian knot in $\mathcal{L}(K_1 \# K_2)$ is equivalent to $L \# L_2$ for some $L_2 \in \mathcal{L}(K_2)$ by moving all stabilizations to the second summand. Therefore, the classical invariants for $L \# L_2$ determine not only the classical invariants for L_2 , but also L_2 itself since K_2 is Legendrian simple. This implies the Legendrian simplicity for $K_1 \# K_2$.

The number of peaks directly follows from Corollary 8. \square

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